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LETTER TO THE EDITOR

Dilute bond Ising model and percolation

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Abstract. A formulation of the dilute bond Ising model is derived in terms of an equivalent Hamiltonian. A temperature concentration phase diagram is proposed. It is shown that, although no spin-glass is expected, the confluence at $T=0$ of the respective conditions for ferromagnetic and spin-glass transitions to occur induces the low temperature crossover to percolation behaviour.

In this letter we propose to point out some characteristic features of the dilute bond Ising model, within the framework of the method developed by Edwards and Anderson (1975) to deal with the spin-glass problem.

The original Hamiltonian of the system is

$$H=\sum_{(i,j)}-J_{ij}S_iS_j \qquad (S=\pm 1)$$

where the summation subscript runs over all pairs of nearest neighbours; each of the N lattice sites has z nearest neighbours; the J_{ij} are $Nz/2$ independent bivalued random variables characterized by

$$\Pr\{J_{ij}=J\}=c$$

$$\Pr\{J_{ij}=0\}=1-c.$$

In what follows, $\langle\ldots\rangle$ denotes thermal averaging for a given distribution of the J_{ij}, whereas a bar denotes a stochastic averaging over the J_{ij}.

We are led to evaluate

$$\overline{\ln Z}=\lim_{n\to 0}\frac{\overline{Z^n}-1}{n}=\lim_{n\to 0}\frac{\overline{\mathrm{Tr}\exp(\beta\sum_{(i,j)}J_{ij}\sum_{\alpha=1}^{n}\sigma_i^\alpha\sigma_j^\alpha)}-1}{n}$$

where, following Edwards and Anderson (1975), n replicas $\sigma_i^\alpha=\pm 1$ of each original S_i spin have been introduced. We have, in a straightforward manner,

$$\overline{Z^n}=\mathrm{Tr}\prod_{(i,j)}\left[1-c+c\exp\left(\beta J\sum_\alpha\sigma_i^\alpha\sigma_j^\alpha\right)\right] \tag{1}$$

which defines an equivalent Hamiltonian $\mathscr{H}$ through

$$-\beta\mathscr{H}=\sum_{(i,j)}\ln\left[1-c+c\exp\left(\beta J\sum_\alpha\sigma_i^\alpha\sigma_j^\alpha\right)\right]. \tag{2}$$

A systematic expansion of $-\beta\mathscr{H}$ in the form

$$-\beta\mathscr{H}=\sum_{(i,j)}\left(\tilde{J}_0+\tilde{J}_1\sum_{\alpha}\sigma_i^\alpha\sigma_j^\alpha+\tilde{J}_2\sum_{\alpha<\beta}\sigma_i^\alpha\sigma_j^\alpha\sigma_i^\beta\sigma_j^\beta+\ldots+\tilde{J}_n\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2\ldots\sigma_i^n\sigma_j^n\right) \tag{3}$$

can be achieved by making use of the following property of a binomial random variable X_b: if $\Pr\{X_b=k\}=\binom{b}{k}p^kq^{b-k}$ then the expected value of $\exp(aX_b)$ is $E[\exp(aX_b)]=(pe^a+q)^b$ which behaves like $1+b\ln(pe^a+q)$ when the analytic continuation for $b\to 0$ is taken out. Conversely, if one puts $p=c$, $q=1-c$:

$$-\beta\mathscr{H}=\sum_{(i,j)}\lim_{b\to 0}\frac{E[\exp(\beta J\sum_\alpha\sigma_i^\alpha\sigma_j^\alpha X_b)]-1}{b}$$

$$=\sum_{(i,j)}\lim_{b\to 0}\frac{1}{b}E\left[[\cosh(\beta JX_b)]^n\prod_\alpha[1+\tanh(\beta J\sigma_i^\alpha\sigma_j^\alpha X_b)]-1\right].$$

Thus, $\tilde{J}_l$ as defined in (3) becomes, for $l\geqslant 1$:

$$\tilde{J}_l=\lim_{b\to 0}\frac{1}{b}E[[\cosh(\beta JX_b)]^n[\tanh(\beta JX_b)]^l].$$

Taking first the limit $n\to 0$, one has explicitly

$$\tilde{J}_l=\lim_{b\to 0}\frac{1}{b}\sum_{k=1}^{k=b}\binom{b}{k}c^k(1-c)^{b-k}[\tanh(\beta Jk)]^l. \tag{4}$$

After expansion of $\tanh(\beta Jk)$ in terms of $\exp(-2\beta Jk)$ and taking the limit $b\to 0$ one finds, for $c<1$:

$$\tilde{J}_0=0$$

$$\tilde{J}_1=-\ln(1-c)+2\sum_{r=1}^{\infty}(-1)^r\ln\left(1+\frac{c}{1-c}\exp(-2\beta Jr)\right)$$

$$\tilde{J}_2=-\ln(1-c)+4\sum_{r=1}^{\infty}(-1)^r r\ln\left(1+\frac{c}{1-c}\exp(-2\beta Jr)\right)$$

$$\tilde{J}_l=-\ln(1-c)+\sum_{r=1}^{\infty}(-1)^r\mathscr{P}_{l-1}(r)\ln\left(1+\frac{c}{1-c}\exp(-2\beta Jr)\right)$$

where $\mathscr{P}_{l-1}(r)$ is a polynomial in r with degree $l-1$. (If $c=1$, all $\tilde{J}_l$ are zero except $\tilde{J}_1=\beta J$; in the vicinity of $c=1$, direct resort to (4) is useful although the absolute convergence of the above series defining the $\tilde{J}_l$ is ensured for all values of c strictly inferior to 1.)

In particular, at $T=0$, all the $\tilde{J}_l$ become equal to $-\ln(1-c)$ (except $\tilde{J}_0=0$), and $-\beta\mathscr{H}$ reduces to the compact form

$$(-\beta\mathscr{H})_{T=0}=\sum_{(i,j)}\ln(1-c)\left(1-\prod_\alpha(1+\sigma_i^\alpha\sigma_j^\alpha)\right)$$

which yields the cluster generating function in the percolation problem.

Due to the complicated form of $\mathscr{H}$ as defined in (3), we now turn to a variational approach of the problem by means of the trial Hamiltonian $\mathscr{H}_0$ defined by

$$\beta\mathscr{H}_0=\sum_i\left[-h\sum_\alpha\sigma_i^\alpha-k\left(\sum_\alpha\sigma_i^\alpha\right)^2\right]$$

which conduces to the possible order parameters $m=\langle\sigma_i^\alpha\rangle$ and $q=\langle\sigma_i^\alpha\sigma_i^\beta\rangle$ as in Edwards and Anderson (1975). (An equivalent expression of m and q in terms of the original Hamiltonian is given by $m=\overline{\langle S_i\rangle}$, $q=\overline{\langle S_i\rangle^2}$).

The trace on the eigenstates of $\mathscr{H}_0$ will be performed by means of the following representation:

$$\exp(-\beta\mathscr{H}_0)=\prod_i\exp\left[h\sum_\alpha\sigma_i^\alpha+k\left(\sum_\alpha\sigma_i^\alpha\right)^2\right]$$
$$=\prod_i\left(\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\ \mathrm{e}^{-\lambda^2/4}\prod_\alpha\exp[(h+\sqrt{k}\lambda)\sigma_i^\alpha]\right)$$
$$=\prod_i\left(\int\mathrm{d}G_\lambda\prod_\alpha\exp[(h+\sqrt{k}\lambda)\sigma_i^\alpha]\right),$$

where $\int\mathrm{d}G_\lambda$ indicates the measure on the normalized Gaussian law. Then, taking the limit $n\to0$,

$$m=\int\mathrm{d}G_\lambda\tanh(h+\sqrt{k}\lambda)$$
$$q=\int\mathrm{d}G_\lambda\tanh^2(h+\sqrt{k}\lambda) \qquad (5)$$

and the variational free energy per site is defined by

$$Nn\beta\tilde{f}=\langle\beta\mathscr{H}\rangle_0-\langle\beta\mathscr{H}_0\rangle_0+Nn\beta f_0$$

with

$$\beta f_0=-\int\mathrm{d}G_\lambda\ln[2\cosh(h+\sqrt{k}\lambda)].$$

Using equations (4) and (5), we expand $\beta\tilde{f}$ up to fourth order in h and k and find (apart from a constant term):

$$\beta\tilde{f}=\tfrac{1}{2}(1-z\tilde{J}_1)h^2-(1-z\tilde{J}_2)k^2+(2z\tilde{J}_1+z\tilde{J}_2-2)h^2k+8(\tfrac{2}{3}-z\tilde{J}_2)k^3$$
$$+(\tfrac{1}{3}z\tilde{J}_1+\tfrac{1}{4}z\tilde{J}_2-\tfrac{1}{4})h^4+(\tfrac{184}{3}z\tilde{J}_2-34)k^4-2(5z\tilde{J}_1+6z\tilde{J}_2+12z\tilde{J}_3-6)h^2k^2. \qquad (6)$$

The paramagnetic–ferromagnetic and paramagnetic–spin-glass transition conditions are respectively given by $z\tilde{J}_1=1$ which defines $T_\mathrm{F}(c)$ and $z\tilde{J}_2=1$ which defines $T_\mathrm{SG}(c)$. It is found that, except at $T=0$ where both critical points are located at $c_\mathrm{p}=1-\exp(-1/z)$ since $\tilde{J}_{1(T=0)}=\tilde{J}_{2(T=0)}=-\ln(1-c)$, $T_\mathrm{F}(c)$ is always greater than $T_\mathrm{SG}(c)$. Therefore, if $T<T_\mathrm{F}(c)$, the system becomes ferromagnetic and no spin-glass phase can be observed (no possibility of a first-order transition has been found by direct inspection of $\beta\tilde{f}$). The corresponding transition curve is plotted in figure 1.

These results call for the following comments.

(*a*) When $T_\mathrm{F}\neq0$, the mean-field behaviour of m and q in the vicinity of the critical line is given by

$$m\sim[A(T_\mathrm{F}-T)+B(c-c_0)]^{1/2}\qquad\text{and}\qquad q\sim[A(T_\mathrm{F}-T)+B(c-c_0)].$$

The critical properties in this case have been discussed by several authors (Khmelnitskii 1975, Grinstein and Luther 1976). In particular, in view of the use of renormalization techniques, the critical dimension is $d_\mathrm{c}=4$.

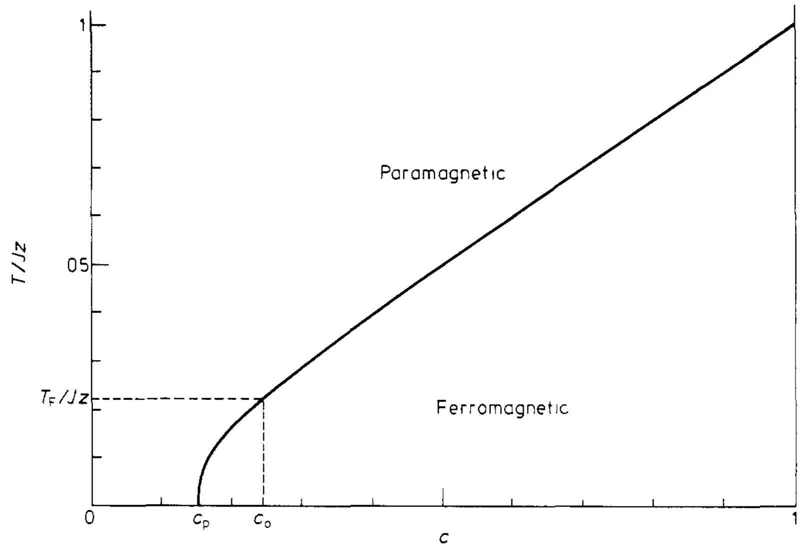

Figure 1. Temperature–concentration phase diagram, computed for $z = 6$.

(*b*) *At* $T = 0$, the dilute Ising model is known to be closely related to the percolation problem (Essam 1972, Dunn *et al* 1975). The critical concentration in the Ising model is equal to the percolation concentration c_p (our mean-field value for c_p ($c_p = 1 - \exp(-1/z)$) has already been obtained by Mittag and Stephen (1974) and Stephen (1976) through an equivalence with the Potts model); the spontaneous magnetization is equal to the so called percolation probability P_p.

Our result shows that the confluence of the two critical points (ferromagnetic and spin-glass) induces an important change in the mean-field behaviour of m and q in comparison with $T \neq 0$. From (6) one finds

$$m = q \sim c - c_p \qquad \text{in the vicinity of } c_p.$$

The equality between m and q is due to the fact that $\langle S_i \rangle = \langle S_i \rangle^2 = 0$ or 1 in any configuration when $T = 0$ (an infinitesimal field being applied in the positive direction).

This result is in agreement with the predictions of the Potts model. The critical dimension in this case is $d_c = 6$; expansions in $6 - d$ of the critical exponents have been performed by Harris *et al* (1975) and Amit (1976).

(*c*) It is now clear that the properties of the system in the vicinity of $T = 0$, $c = c_p$ are dominated by a crossover phenomenon between the two behaviours described in (*a*) and (*b*), as discussed by Stauffer (1975). To make this point explicit let us write

$$\tilde{J}_2 = \tilde{J}_1(1 + \Delta) \qquad \text{where } \Delta \sim e^{-2J/T} \text{ when } T \to 0.$$

This is a case of quadratic anisotropy (Fisher and Pfeuty 1972). (Note that, in our mean-field approximation, $T_F(c)$ is defined by $(c - c_p)/2c_p = e^{-2J/T_F}$ when $c \to c_p$.)

General crossover scaling predictions claim that (using the standard notations):

$$F_{\text{sing}}(m, \tau, \Delta) \sim |\tau|^{2-\alpha_p} f\left(\frac{m}{|\tau|^{\beta_p}}, \frac{\Delta}{|\tau|^{\phi}}\right)$$

$$\Gamma(x, \tau, \Delta) \sim x^{2-(d+\eta_p)} g\left(\frac{x}{|\tau|^{-\nu_p}}, \frac{\Delta}{|\tau|^{\phi}}\right)$$

where F_{sing} is the singular part of the free energy, Γ the two-point correlation function, and $\tau = (\tilde{J}_{1c}(\Delta) - \tilde{J}_1)/\tilde{J}_1 \sim c - c_0$ is defined by analogy with the usual anisotropy problems; α_p, β_p, η_p, ν_p are the critical indices of percolation.

The displacement of the critical concentration with T is given by

$$\tau_c = \frac{\tilde{J}_{1c}(0) - \tilde{J}_{1c}(\Delta)}{\tilde{J}_{1c}(\Delta)} \sim c_0 - c_p - a\Delta \sim \Delta^{1/\psi}.$$

If $\psi > 1$ as usual, then

$$c_0 - c_p \sim \Delta^{1/\psi} \sim e^{-2J/\psi T}$$

and the crossover line between random Ising model and percolation behaviours is evaluated by

$$c_0 - c^* \cong e^{-2J/\phi T}.$$

These results confirm Stauffer's *ansatz* (Stauffer 1975) concerning the measure of the 'temperature' by the quantity $e^{-2J/T}$.

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